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Fermion generations from "apple-shaped" extra dimensions

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Abstract: We examine the behavior of fermions in the presence of an internal compact 2-manifold which in one of the spherical angles exhibits a conical character with an obtuse angle. The extra manifold can be pictured as an apple-like surface i.e. a sphere with an extra 'wedge' insert. Such a surface has conical singularities at north and south poles. It is shown that for this setup one can obtain, in four dimensions, three trapped massless fermion modes which differ from each other by having different values of angular momentum with respect to the internal 2-manifold. The extra angular momentum acts as the family label and these three massless modes are interpreted as the three generations of fundamental fermions.

Keywords: Flux compactifications, p-branes.

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1. Introduction

One of the open questions in the Standard Model of particle physics is the fermion family puzzle - why the first generation of quarks and leptons are replicated in two other families of increasing mass. It is not clear how to explain the mass hierarchy of the generations and the mixing between the families characterized by the Cabbibo-Kobayashi-Maskawa matrix. Several ideas have been suggested such as a horizontal family symmetry [1].

Recently the brane world idea [2] has been used to find new solutions to old problems in particle physics and cosmology. A key requirement for theories with extra dimensions is that the various bulk fields be localized on the brane. Brane solutions with different matter localization mechanisms have been widely investigated in the scientific literature [3]. A pure *gravitational* trapping of zero modes of all bulk fields was given in [4, 5].

The main emphasis of the present paper is to explain some properties of fermion families in the framework of a brane model. For the other attempts using extra dimensions see [6, 7]. We introduce an extra 2-dimensional compact surface and investigate the properties of higher dimensional fermions place in this space-time. In 6-dimensional models the internal compact 2-manifold usually is considered as having rugby (football)-ball shaped geometry with a deficit angle [8, 9]. As shown in this paper one can address the generation puzzle using an internal 2-surface with a profuse angle, or having an "apple-like" geometry. Using the brane solution of [5] we show that for apple-shaped extra dimensions three fermion generations naturally arise from the zero modes of a *single* 6-dimensional spinor field. This gives a purely geometrical mechanism for the origin of three generations of the Standard Model fermions from one generation in a higher-dimensional theory. The localized fermions are stuck at different points in the extra space similar to the model [7]. A mass hierarchy and mixings between the three zero modes are obtained by introducing of a Yukawa-type coupling to a single 6-dimensional scalar field.

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2. Solution of 6-dimensional Einstein equations

In this article we consider 6-dimensional space-time with the signature $(+----{-}{-})$. Einstein's equations in this space have the form

$$R_{AB} - \frac{1}{2} g_{AB} R = \frac{1}{M^4} \left(g_{AB} \Lambda + T_{AB} \right) , \tag{2.1}$$

where M and Λ are the 6-dimensional fundamental scale and the cosmological constant. Capital Latin indices run over $A, B, \ldots = 0, 1, 2, 3, 5, 6$.

To split the space-time into 4-dimensional and 2-dimensional parts we use the metric *ansatz*

$$ds^2 = \phi^2 (\theta)\, g_{\mu\nu} (x^\alpha)\, dx^\mu dx^\nu - \varepsilon^2 \left(d\theta^2 + b^2 \sin^2 \theta d\varphi^2 \right) , \tag{2.2}$$

where ε and b are constants. Here the metric of ordinary 4-space-time, $g_{\mu\nu} (x^\alpha)$, has the signature $(+---)$ (the Greek indices $\alpha, \mu, \nu \ldots = 0, 1, 2, 3$ refer to 4-dimensional coordinates). The extra compact 2-manifold is parameterized by the two spherical angles θ and φ ($0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi$). We take this 2-surface to be attached to the brane at the point $\theta = 0$. Thus the geodesic distance into the extra dimensions goes from north to south pole of the extra 2-spheroid when θ changes from 0 to π.

If in (2.2) the constant $b = 1$ then the extra 2-surface is exactly a 2-sphere with the radius ε. If $b \neq 1$ the extra manifold is a 2-spheroid with either a deficit or profuse angle φ, i.e. its conical sections, $\theta = const$, are either missing some angle, $\delta\varphi$, or have some extra angle, $\delta\varphi$. The metric for this 2-manifold will take usual form with $b = 1$ if we redefine φ so it ranges from 0 to $2\pi b$. One can think of the extra 2-surface as being of sphere with cut out (if $b < 1$), or inserted (if $b > 1$) the "wedge" having an angle $\delta\varphi = 2\pi(b-1)$. This gives a δ-like contribution to the curvature tensor localized at the points with $\sin\theta = 0$. These singularities can be cancelled by introduction of 3-branes at these positions [8]. Usually in the literature one considers the case $b < 1$ with the deficit angle leading to rugby(football)-ball shaped geometry [9]. As it will be clear below we need the case $b > 1$ which gives a profuse angle. Thus the extra 2-manifold can be imagined as the apple-like surface.

The *ansatz* for the energy-momentum tensor of the bulk matter fields we take in the form

$$T_{\mu\nu} = -g_{\mu\nu} E (\theta) , \qquad T_{ij} = -g_{ij} P (\theta) , \qquad T_{i\mu} = 0 , \tag{2.3}$$

small Latin indices correspond to the two extra coordinates. The source functions E and P depend only on the extra coordinate θ.

For these *ansätze* Einstein's equations (2.1) take the following form:

$$\begin{aligned} 3\frac{\phi''}{\phi} + 3\frac{\phi'^2}{\phi^2} + 3\frac{\phi'}{\phi}\cot\theta - 1 &= \frac{\varepsilon^2}{M^4} \left[E (\theta) - \Lambda \right] , \\ 6\frac{\phi'^2}{\phi^2} + 4\frac{\phi'}{\phi}\cot\theta &= \frac{\varepsilon^2}{M^4} \left[P (\theta) - \Lambda \right] , \\ 4\frac{\phi''}{\phi} + 6\frac{\phi'^2}{\phi^2} &= \frac{\varepsilon^2}{M^4} \left[P (\theta) - \Lambda \right] , \end{aligned} \tag{2.4}$$

where the prime denotes differentiation $d/d\theta$. For the 4-dimensional space-time we have assumed zero cosmological constant and Einstein's equations in the form

$$R^{(4)}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{(4)} = 0, \tag{2.5}$$

where $R^{(4)}_{\alpha\beta}$ and $R^{(4)}$ are 4-dimensional Ricci tensor and scalar curvature.

In [5] a non-singular solution of (2.4) was found for the boundary conditions $\phi(0) = 1,\ \phi'(0) = 0$. The solution was given by

$$\phi\,(\theta) = 1 + (a-1)\sin^2(\theta/2)\,, \tag{2.6}$$

where a is the integration constant. The source terms for this solution are

$$E(\theta) = \Lambda\left[\frac{3(a+1)}{5\phi(\theta)} - \frac{3a}{10\phi^2(\theta)}\right], \quad P(\theta) = \Lambda\left[\frac{4(a+1)}{5\phi(\theta)} - \frac{3a}{5\phi^2(\theta)}\right] \tag{2.7}$$

and the radius of the extra 2-spheroid given by $\varepsilon^2 = 10M^4/\Lambda$. For simplicity in this paper we take $a = 0$ so that below we will use the warp factor

$$\phi\,(\theta) = 1 - \sin^2(\theta/2) = \cos^2(\theta/2)\ . \tag{2.8}$$

This warp factor equals one at the brane location ($\theta = 0$) and decreases to zero in the asymptotic region $\theta = \pi$, i.e. at the south pole of the extra 2-dimensional spheroid.

The expression for the determinant of our *ansatz* (2.2), which will be used often in what follows, is given by

$$\sqrt{-g} = \sqrt{-g^{(4)}}\varepsilon^2\phi^4(\theta)\sin\theta\,, \tag{2.9}$$

where $\sqrt{-g^{(4)}}$ is determinant of 4-dimensional space-time.

3. Fermions in six dimension

Let us consider spinors in the 6-dimensional space-time (2.2), where the warp factor $\phi(\theta)$ has the form (2.8). The action integral for the 6-dimensional massless fermions in a curved background is

$$S_\Psi = \int d^6x\sqrt{-g}\left[i\overline{\Psi}h^B_{\widetilde{A}}\Gamma^{\widetilde{A}}D_B\Psi + h.c.\right], \tag{3.1}$$

D_A denote covariant derivatives, $\Gamma^{\widetilde{A}}$ are the 6-dimensional flat gamma matrices and we have introduced the *sechsbein* $h^{\widetilde{A}}_A$ through the usual definition

$$g_{AB} = h_A{}^{\widetilde{A}}h_B{}^{\widetilde{B}}\eta_{\widetilde{A}\widetilde{B}}\,, \tag{3.2}$$

$\widetilde{A}, \widetilde{B}, \ldots$ are local Lorentz indices.

In six dimensions a spinor

$$\Psi(x^A) = \begin{pmatrix}\psi\\ \xi\end{pmatrix} \tag{3.3}$$

has eight components and is equivalent to a pair of 4-dimensional Dirac spinors, ψ and ξ.

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In this paper we use the following representation of the flat (8×8) gamma-matrices (for simplicity we drop the tildes on the local Lorentz indices when no confusion will occur)

$$\Gamma_\nu = \begin{pmatrix} \gamma_\nu & 0 \\ 0 & -\gamma_\nu \end{pmatrix}, \qquad \Gamma_\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_\varphi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3.4}$$

where 1 denotes the 4-dimensional unit matrix and γ_ν are ordinary (4×4) gamma-matrices. It is easy to check that the representation (3.4) gives the correct space-time signature $(+-----)$. The 6-dimensional analog of the γ_5 matrix in the representation (3.4) has the form

$$\Gamma_7 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix} . \tag{3.5}$$

From (3.3) one finds that the 6-dimensional left-handed (right-handed) particles correspond to a pair of 4-dimensional particles, ψ and ξ which correspond to the particle (anti-particle) from the 4-dimensional point of view.

The 6-dimensional massless Dirac equation, which follows from the action (3.1), has the form

$$\left(h^\mu_{\widetilde{B}}\Gamma^{\widetilde{B}}D_\mu + h^\theta_{\widetilde{B}}\Gamma^{\widetilde{B}}D_\theta + h^\varphi_{\widetilde{B}}\Gamma^{\widetilde{B}}D_\varphi\right)\Psi(x^A) = 0\,, \tag{3.6}$$

with the *sechsbein* for our background metric (2.2) given by

$$h_{\widetilde{A}}^{\ B} = \left(\frac{1}{\phi}\delta^B_{\widetilde{\mu}},\ \frac{1}{\varepsilon}\delta^B_{\widetilde{\theta}},\ \frac{1}{b\varepsilon\sin\theta}\delta^B_{\widetilde{\varphi}}\right) . \tag{3.7}$$

From the definition

$$\begin{aligned}\omega_M^{\widetilde{M}\widetilde{N}} = \frac{1}{2}h^{N\widetilde{M}}\left(\partial_M h^{\widetilde{N}}_N - \partial_N h^{\widetilde{N}}_M\right) - \frac{1}{2}h^{N\widetilde{N}}\left(\partial_M h^{\widetilde{M}}_N - \partial_N h^{\widetilde{M}}_M\right) \\ -\frac{1}{2}h^{P\widetilde{M}}h^{Q\widetilde{N}}\left(\partial_P h_{Q\widetilde{R}} - \partial_Q h_{P\widetilde{R}}\right)h^{\widetilde{R}}_M\end{aligned} \tag{3.8}$$

the non-vanishing components of the spin-connection for the *sechsbein* (3.7) can be found

$$\omega_\varphi^{\widetilde{\theta}\widetilde{\varphi}} = -b\cos\theta, \qquad \omega_\nu^{\widetilde{\theta}\widetilde{\nu}} = -\frac{\phi'}{\varepsilon} = \frac{\sin\theta}{2\varepsilon} . \tag{3.9}$$

The covariant derivatives of the spinor field have the forms

$$\begin{aligned} D_\mu\Psi(x^A) &= \left[\partial_\mu + \frac{\sin\theta}{4\varepsilon}\Gamma_\theta\Gamma_\nu\right]\Psi(x^A)\ , \\ D_\theta\Psi(x^A) &= \partial_\theta\Psi(x^A)\ , \\ D_\varphi\Psi(x^A) &= \left(\partial_\varphi - \frac{b\cos\theta}{2}\Gamma_\theta\Gamma_\varphi\right)\Psi(x^A)\ . \end{aligned} \tag{3.10}$$

Then Dirac's equation (3.6) takes the form [10, 11]

$$\begin{aligned}\left[\frac{1}{\phi}\Gamma^\mu\frac{\partial}{\partial x_\mu} + \frac{\sin\theta}{4\varepsilon\phi}\Gamma^\nu\Gamma_\theta\Gamma_\nu + \frac{1}{\varepsilon}\Gamma^\theta\frac{\partial}{\partial\theta} + \frac{1}{b\varepsilon\sin\theta}\Gamma^\varphi\frac{\partial}{\partial\varphi} - \frac{\cot\theta}{2\varepsilon}\Gamma^\varphi\Gamma_\theta\Gamma_\varphi\right]\Psi(x^A) = \\ \left[\frac{1}{\phi}\Gamma^\mu\frac{\partial}{\partial x_\mu} + \frac{1}{\varepsilon}\Gamma^\theta\left(\frac{\partial}{\partial\theta} - \frac{\sin\theta}{\phi} + \frac{\cot\theta}{2}\right) + \frac{1}{b\varepsilon\sin\theta}\Gamma^\varphi\frac{\partial}{\partial\varphi}\right]\Psi(x^A) = 0\ .\end{aligned} \tag{3.11}$$

This system of first order partial differential equations can be treated using the following separation of variables

$$\Psi(x^A) = \sum_l \frac{e^{il\varphi}}{\sqrt{2\pi}\phi^2(\theta)} \begin{pmatrix} \alpha_l(\theta)\psi_l(x^\nu) \\ \beta_l(\theta)\xi_l(x^\nu) \end{pmatrix}, \tag{3.12}$$

where $\psi_l(x^\nu)$ and $\xi_l(x^\nu)$ are 4-dimensional Dirac spinors. Here we note that since dimension of $\Psi(x^A)$ in six dimensions is $m^{5/2}$ then dimensions of $\alpha_l(\theta)$, $\beta_l(\theta)$ and $\psi_l(x^\nu)$, $\xi_l(x^\nu)$ should be m and $m^{3/2}$ respectively.

At the end of the section we note that our case is unlike the model studied in [11], which examined spin-1/2 particles confined on a 2-sphere. In our case the internal 2-manifold is only a part of the bulk 6-dimensional space-time and we are looking for spinors in four dimensions. It is the functions $\psi_l(x^\nu)$ and $\xi_l(x^\nu)$ in (3.12) which must have spinor representations. So the wave-function given in (3.12) is single-valued for 2π rotations around the brane by the extra angle φ. Thus the quantum number l takes integer values — $l = 0, \pm 1, \pm 2, \ldots$ — and not half-integer values.

4. Fermion generations

We are looking for 4-dimensional fermionic zero modes. To this end we take $\psi_l(x^\nu)$ and $\xi_l(x^\nu)$ in (3.12) to obey the 4-dimensional, massless Dirac equations

$$\gamma^\mu \partial_\mu \psi_l(x^\nu) = \gamma^\mu \partial_\mu \xi_l(x^\nu) = 0 \ . \tag{4.1}$$

There will also be very massive KK modes whose masses will go a integer multiples of the inverse size of the extra 2-dimensional space i.e. as $1/\varepsilon$. However, we will assume later that $1/\varepsilon \simeq 1\,\text{TeV}$. Thus these massive KK modes have a much higher mass and are distinct from the three fermion generations. For the massless case the 4-spinors $\psi_l(x^\nu)$ and $\xi_l(x^\nu)$ are indistinguishable from the 4-dimensional point of view and we can write

$$\psi_l(x^\nu) = \xi_l(x^\nu) \ . \tag{4.2}$$

Inserting (3.12), (4.1) and (4.2) into (3.11) converts the bulk Dirac equation into

$$\left[\Gamma^\theta \left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}\right) + \frac{il}{b\sin\theta}\Gamma^\varphi\right] \begin{pmatrix} \alpha_l(\theta) \\ \beta_l(\theta) \end{pmatrix} = 0 \ . \tag{4.3}$$

Using the representation for Γ^θ, Γ^φ gives the following system of equations for $\alpha_l(\theta)$ and $\beta_l(\theta)$

$$\begin{aligned} \left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} - \frac{l}{b\sin\theta}\right)\alpha_l(\theta) &= 0\,, \\ \left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2} + \frac{l}{b\sin\theta}\right)\beta_l(\theta) &= 0\ . \end{aligned} \tag{4.4}$$

The solutions of these equations are

$$\alpha_l(\theta) = A_l \frac{\tan^{l/b}(\theta/2)}{\sqrt{\sin\theta}}\,, \qquad \beta_l(\theta) = B_l \frac{\tan^{-l/b}(\theta/2)}{\sqrt{\sin\theta}}\,, \tag{4.5}$$

where A_l and B_l are integration constants with the dimension of mass.

The normalizable modes are those for which

$$\int \sqrt{-g}\; d^6x\; \bar{\Psi}\Psi = \int \sqrt{g^{(4)}}\; d^4x\; \left(\bar{\psi}_l\psi_l + \bar{\xi}_l\xi_l\right)\;. \tag{4.6}$$

In other words we want the integral over the extra coordinates, φ and θ, to equal 1. Thus inserting (3.12), (4.5) and the determinant (2.9) into (4.6) the requirement that the integral over φ and θ equal 1 gives

$$\varepsilon^2 \int_0^\pi d\theta \left[A_l^* A_l \tan^{2l/b}(\theta/2) + B_l^* B_l \tan^{-2l/b}(\theta/2)\right] = 1\,, \tag{4.7}$$

where the integral over φ contributes 2π.

Using the formula

$$\int_0^\pi d\theta\;\; \tan^{2c}(\theta/2) = \frac{\pi}{\cos(c\pi)}\,, \qquad\qquad -1 < 2c < 1 \tag{4.8}$$

we see that (4.7) is convergent only for the case

$$-b < 2l < b\;. \tag{4.9}$$

Recall that the parameter b in (2.2) is an integration constant of Einstein's equations and governs the topology of the internal 2-spheroid. If $b = 1$ the internal 2-surface is exactly a sphere. For this case, as it clear from (4.9), there exist only one zero mode with $l = 0$. If on the other hand $2 < b \leq 4$ we have exactly three fermionic zero modes with the quantum numbers $l = 0$ and $l = \pm 1$. To be concrete we will set $b = 4$ in the following. Other choices of b from this interval will only slightly change the numerical results below. From the normalization condition (4.7) we now find the following relation for the constants A_l and B_l

$$\pi\varepsilon^2(A_l^* A_l + B_l^* B_l) = \cos\frac{l\pi}{4}\,, \tag{4.10}$$

where $l = 0, \pm 1$.

Explicitly the expressions for the three normalizable 8-spinors (3.12) that solve the 6-dimensional Dirac equations (3.11) are

$$\begin{aligned}
\Psi_0(x^A) &= \frac{1}{\sqrt{2\pi\sin\theta}\;\phi^2(\theta)} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \psi_0(x^\nu)\,, \\
\Psi_1(x^A) &= \frac{1}{\sqrt{2\pi\sin\theta}\;\phi^2(\theta)}\; e^{i\varphi} \begin{pmatrix} \tan^{1/4}(\theta/2)A_1 \\ \tan^{-1/4}(\theta/2)B_1 \end{pmatrix} \psi_1(x^\nu)\,, \\
\Psi_{-1}(x^A) &= \frac{1}{\sqrt{2\pi\sin\theta}\;\phi^2(\theta)}\; e^{-i\varphi} \begin{pmatrix} \tan^{-1/4}(\theta/2)A_{-1} \\ \tan^{1/4}(\theta/2)B_{-1} \end{pmatrix} \psi_{-1}(x^\nu)\,,
\end{aligned} \tag{4.11}$$

where the constants A_l and B_l obey the relations (4.10).

These three normalizable modes all appear as massless 4-dimensional fermions on the brane. To explain the observed mass spectrum and mixing between these fermions one needs to couple these particles to a scalar (Higgs) field.

5. Coupling with Higgs field

In the previous section it was shown that by adjusting the integration constant b in our gravitational background (2.2) it is possible to get three zero-mass modes on the brane. To make this model more realistic we have two problems:

a) There is no mixing between the different generations due to the orthogonality of the angular parts of the higher dimensional wave functions. Overlap integrals like $\int d\varphi\ \bar{\psi}_l \psi_{l'}$, which characterize the mixing between the different states, vanish since

$$\int_0^{2\pi} d\varphi\ e^{-il\varphi} e^{il'\varphi} = 0\ . \qquad l \neq l' \tag{5.1}$$

b) All the fermionic states (4.11) are massless, whereas the fermions of the real world have masses that increase with each family.

Following [12] we address both of these issues by introducing a coupling between the fermions with the bulk scalar field $\Phi_p(x^A)$ (which has dimensions m^2) by adding to the action an interaction term of the form

$$S_{\rm int} = \frac{1}{F} \int d^4x d\varphi d\theta \sqrt{-g}\ \Phi_p \bar{\Psi}_l \Psi_{l'}\ , \tag{5.2}$$

where F is the coupling constant between the scalar and spinor fields and has the dimension of mass.

For simplicity we take the massless, real scalar field to be of the form

$$\Phi_p(x^A) = \kappa_p\ \Phi_p(\theta)\ e^{ip\varphi}\ , \tag{5.3}$$

i.e. the scalar field only depends on the bulk coordinates θ, φ, not on the brane coordinates x^μ. In (5.3) the angular quantum number p is an integer and κ_p are the 4-dimensional constant parts of $\Phi_p(x^A)$ having dimensions of mass.

The equation of motion of a massless real scalar field in six dimensions has the form:

$$\frac{1}{\sqrt{-g}} D_A \left[\sqrt{-g}\ g^{AB}\ D_B \Phi(x^A)\right] = 0\ . \tag{5.4}$$

Using the form of Laplace operator on our 2-spheroid

$$\Delta_2 = -\frac{1}{\varepsilon^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{4\phi'}{\phi}\frac{\partial}{\partial\theta} + \frac{1}{b^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\ , \tag{5.5}$$

where ϕ is given by (2.8), the equation (5.4) can be written as

$$\Phi_p'' + \left(\cot\theta - \frac{4\sin\theta}{1+\cos\theta}\right)\Phi_p' - \frac{p^2}{b^2\sin^2\theta}\Phi_p = 0\ . \tag{5.6}$$

It is possible to give an exact solution to this equation in quadratures. However, this solution is a complicated function. In order to make understandable estimates of the masses and mixings we will use approximate solutions. Close to the origin ($\theta \to 0$), when $\sin\theta \to 0$ and $\phi \to 1$ this equation can be approximated as

$$\Phi_p'' + \cot\theta \Phi_p' - \frac{p^2}{b^2\sin^2\theta}\Phi_p = 0\ . \tag{5.7}$$

For $p = 0$ a solution to this equation is

$$\Phi_0(\theta) = D_0 \left\{1 + \ln\left[\tan(\theta/2)\right]\right\} , \qquad p = 0 \tag{5.8}$$

where D_0 is an integration constant.

For non-zero p one of the solutions of (5.7) is

$$\Phi_p(\theta) = D_p \cosh\left\{\frac{p}{b} \ln\left[\cot(\theta/2)\right]\right\} , \qquad p \neq 0 \tag{5.9}$$

where D_p are integration constants. Note that these solutions (as well as the spinor fields (3.12) and (4.5)) are singular at $\sin\theta = 0$, however, because of the determinant (2.9) the various integrals done with these fields are finite.

We determine the constants D_p by requiring that the scalar modes are normalized over the extra coordinates, i.e. using (2.9) we require

$$2\pi\varepsilon^2 \int_0^\pi d\theta \ \sin\theta \ \phi^4(\theta) \ \Phi_p^2(\theta) = 1 \ . \tag{5.10}$$

For the values of a and b used in this paper ($a = 0$, $b = 4$) from (5.10) we find

$$\begin{aligned} D_0 &= \frac{1}{\varepsilon\sqrt{\frac{\pi^3}{15} - \frac{17\pi}{60}}} \approx \frac{0.92}{\varepsilon} , \\ D_{\pm 1} &= \frac{1}{\varepsilon\sqrt{\frac{2\pi}{5} + \frac{447\sqrt{2}\pi^2}{4096}}} \approx \frac{0.60}{\varepsilon} , \\ D_{\pm 2} &= \frac{1}{\varepsilon\sqrt{\frac{2\pi}{5} + \frac{35\pi^2}{128}}} \approx \frac{0.50}{\varepsilon} \ . \end{aligned} \tag{5.11}$$

Substituting (5.3) and (3.12) into (5.2) we find

$$S_{\rm int} = U^p_{l,l'} \int d^4x \ \sqrt{-g^{(4)}} \ \bar{\psi}_l(x^\mu)\psi_{l'}(x^\mu) , \tag{5.12}$$

with

$$U^p_{l,l'} = \frac{\varepsilon^2 f_p}{2\pi} \int_0^{2\pi} d\varphi e^{i(p-l+l')\varphi} \int_0^\pi d\theta \ \sin\theta \ \Phi_p(\theta) \left[A^*_l A_{l'} \alpha_l(\theta)\alpha_{l'}(\theta) + B^*_l B_{l'} \beta_l(\theta)\beta_{l'}(\theta)\right] , \tag{5.13}$$

where D_p are expressed by (5.11) and A_l, B_l obey the relations (4.10). The constants $f_p = \kappa_p/F$ here denote the ratios of the 4-dimensional constant values of Higgs field from (5.3) and of the coupling constant from (5.1).

The first integral in (5.13) for the quantities $U^p_{ll'}$ will be non-zero if

$$p - l + l' = 0 \ . \tag{5.14}$$

When $l = l'$ and $p = 0$ this gives a mass term; when $l \neq l'$ and $p \neq 0$ this gives mixings between the l and l' modes.

6. Masses and mixings

To find mass terms appearing because of coupling of the three fermionic zero modes (4.11) with the Higgs field (5.3) for the angular momentum quantum numbers in (5.13) we should use the values, $p = 0\,,\ l = l'$, or calculate only the components of the matrix (5.13) with the zero upper index. Using (4.5) and (5.8) from (5.13) we get

$$\begin{aligned} U^0_{0,0} &= f_0 D_0 \varepsilon^2 \pi \left(A_0^* A_0 + B_0^* B_0\right) \\ &= f_0 D_0\,, \\ U^0_{1,1} &= f_0 \frac{D_0 \varepsilon^2 \pi}{\sqrt{2}} \left[(2+\pi) A_1^* A_1 + (2-\pi) B_1^* B_1\right] \\ &= f_0 D_0 \left(\frac{2-\pi}{2} + \sqrt{2}\varepsilon^2 \pi^2 |A_1|^2\right)\,, \\ U^0_{-1,-1} &= f_0 \frac{D_0 \varepsilon^2 \pi}{\sqrt{2}} \left[(2-\pi) A_{-1}^* A_{-1} + (2+\pi) B_{-1}^* B_{-1}\right] \\ &= f_0 D_0 \left(\frac{2+\pi}{2} - \sqrt{2}\varepsilon^2 \pi^2 |A_{-1}|^2\right)\,. \end{aligned} \tag{6.1}$$

To obtain the last equality in each term above we have used (4.10) to eliminate $|B_{\pm 1}|^2$ in favor of $|A_{\pm 1}|^2$.

As a concrete example of how the realistic mass hierarchy can arise let us take $1/\varepsilon \simeq 1\,\mathrm{TeV}$ so that from (5.11) we have $D_0 \simeq 1\,\mathrm{TeV}$. This choice is made so that the massive KK modes (whose mass $\simeq 1/\varepsilon$) will be much heavier than the three zero mass modes, even after they are given a mass via the Higgs mechanism. Next let us examine three quarks from the "down" sector, i.e. d, s and b quarks. This is meant as a toy model since we do not have an "up" sector and we do not have three generations of leptons. Our aim here is just to show that it is possible to generate a realistic fermion mass hierarchy from an extra dimensional model.

Making the association that d-quark $\rightarrow\ l = +1$, s-quark $\rightarrow\ l = -1$ and b-quark $\rightarrow\ l = 0$, we get the following conditions on $U^0_{l,l}$ from (6.1)

$$U^0_{1,1} = m_d \approx 5\ MeV\,, \qquad U^0_{-1,-1} = m_s \approx 100\,MeV\,, \qquad U^0_{0,0} = m_b \approx 4200\ MeV\,, \tag{6.2}$$

where we have taken average values of the quark masses from [13]. Solving the system (6.1) and (6.2) gives

$$f_0 \approx 4.2 \times 10^{-3}\,, \qquad |A_1| \approx 0.202/\varepsilon\,, \qquad |A_{-1}| \approx 0.427/\varepsilon\ . \tag{6.3}$$

Note these values of $|A_{\pm 1}|$ are consistent with the condition in (4.10) which requires $|A_{\pm 1}|,\ |B_{\pm 1}| < \frac{0.474}{\varepsilon}$. The largest mass corresponds to the $l = 0$ quantum number. This can be understood from the point of view that this state has a non-zero effective wavefunction near the brane, $\theta = 0$, and thus has a large overlap with the scalar field (5.8). (By effective wavefunction we mean the combination of the wavefunctions from (4.11) and the square root of the determinant from (2.9). In this way the singular $\sin\theta$ term cancels out). The

d and s quarks, which correspond to the $l = +1, -1$ states, have effective wavefunctions which are zero at $\theta = 0$ and thus have a smaller overlap with the scalar field.

For mixings between the different families, characterized by different angular momentum l, the selection rule (5.14) indicates that we must consider components of the matrix (5.13), which have a nonzero upper index p. There are three independent components whose indices are given by

$$\begin{aligned}
U^1_{1,0} &= U^{-1}_{0,1} \\
&= f_1\varepsilon^2 D_1 \frac{(1+\sqrt{2})\pi}{2}(A_1^* A_0 + B_1^* B_0) \\
&= f_1\varepsilon^2 D_{-1} \frac{(1+\sqrt{2})\pi}{2}(A_0^* A_1 + B_0^* B_1)\,, \\
U^1_{0,-1} &= U^{-1}_{-1,0} \\
&= f_1\varepsilon^2 D_1 \frac{(1+\sqrt{2})\pi}{2}(A_0^* A_{-1} + B_0^* B_{-1}) \\
&= f_1\varepsilon^2 D_{-1} \frac{(1+\sqrt{2})\pi}{2}(A_{-1}^* A_0 + B_{-1}^* B_0)\,, \\
U^2_{1,-1} &= U^{-2}_{-1,1} \\
&= f_2\varepsilon^2 D_2 \sqrt{2}\pi(A_1^* A_{-1} + B_1^* B_{-1}) \\
&= f_2\varepsilon^2 D_{-2} \sqrt{2}\pi(A_{-1}^* A_1 + B_{-1}^* B_1)\ .
\end{aligned} \tag{6.4}$$

From [13] one finds that the mixing between the first and second generation is of order 0.1 (i.e. $V_{us} \approx 0.224$), between the second and third generation of order 0.01 (i.e. $V_{cb} \approx 0.04$), and between the first and third generation of order 0.001 (i.e. $V_{ub} \approx 0.0036$). We take this "up-down" sector mixing as representing generic inter-family mixing, since in our model we have only one flavor in each family (the "down" sector and thus only neutral currents). Then from our previous association of generations (first, second, third) with the internal quantum number l $(+1, -1, 0)$ we arrive at the following connections for the mixings from (6.4)

$$|U^2_{1,-1}| \to V_{us} \simeq 0.1\,, \qquad |U^1_{0,-1}| \to V_{cb} \simeq 0.01\,, \qquad |U^1_{1,0}| \to V_{ub} \simeq 0.001\ . \tag{6.5}$$

In terms of ratios we want to fix A_l, B_l such that from (6.4) we get

$$\frac{|U^1_{1,0}|}{|U^1_{0,-1}|} \simeq 0.1\,, \qquad \frac{|U^1_{0,-1}|}{|U^2_{1,-1}|} \simeq 0.1\ . \tag{6.6}$$

To simplify the analysis we assume that all A_l, B_l are purely real. Then from (4.10) using (6.3) we have

$$|B_1| \approx 0.429/\varepsilon\,, \qquad |B_{-1}| \approx 0.206/\varepsilon\ . \tag{6.7}$$

Also we take $B_0 = kA_0$ where k is some real constant, i.e. from (4.10) B_0 is determined once A_0 is given.

Applying all this to the first condition from (6.6) we find

$$\frac{|U^1_{1,0}|}{|U^1_{0,-1}|} = \frac{0.202 + 0.429k}{0.427 + 0.206k} = 0.1\ . \tag{6.8}$$

Solving for k gives $k = -0.391$. For this value of k we find from (4.10) that

$$A_0 = 0.525/\varepsilon\,, \qquad B_0 = -0.205/\varepsilon\ . \tag{6.9}$$

Inserting all these real values for A_l, B_l into the second condition from (6.6) we find that

$$\frac{|U^1_{0,-1}|}{|U^2_{1,-1}|} = 1.065\ \frac{f_1}{f_2}\ . \tag{6.10}$$

It is clear that if we set $f_1/f_2 \sim 0.1$ (by adjusting κ_1 and κ_2 in (5.3)) we reproduce the mixings between the different generations as given by the rough estimate (6.5).

7. Summary and conclusions

We have given a higher dimensional model to address the fermion generation puzzle. Three zero mass modes arise in an "apple" geometry given by (2.2) and (2.8). Exactly three zero modes are obtained by adjusting the shape of the internal 2-dimensional space via b giving a profuse angle rather than the more common case of a deficit angle. We interpret these three zero mass modes as a toy model for the three generations of fermions. This is a toy model since we do not reproduce the full flavor structure of the Standard Model fermions. For example in this paper we took the three zero mass modes as the down quarks, d, s, b leaving out the up quarks and leptons. The family number in this model was the quantum number l associated with angular momentum of fermions with respect to the extra 2-space.

To give masses and mixings one had to couple the zero mass modes to a scalar field. Thus in this model the masses and mixings arose from the same mechanism. We demonstrated that one could get a realistic mass spectrum and mixings by taking our zero mass modes to be the family of down quarks. That we are able to reproduce a realistic masses and mixings is not surprising since there are number of free parameters involved especially in terms of the normalization constants, κ_p, A_l, B_l for the higher dimensional wavefunctions. The central point of this paper was not so much to obtain a realistic masses and mixings (since in any case the model does not contain complete set of particles of the Standard Model) but rather to give a higher dimensional model for the fermion generation puzzle.

In addition to the zero mass modes there will be massive KK modes whose masses will be of the order $1/\varepsilon$. Here, since $1/\varepsilon \simeq 1\,\text{TeV}$ these massive KK modes would lie well above the three zero mass modes even after they are given masses. In any of these higher dimensional models used to address the generation problem the internal space must be of a small enough size so that the massive KK modes are well separated from the zero mass modes after they are given a mass.

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